The Dependence of Reflection on Incidence Angle*

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Summary—**A lossy dielectric sheet has complex dielectric constant $\epsilon=\epsilon(x)$ and complex permeability $\mu=\mu(x)$, where x is the distance to one interface. This sheet is backed by a conducting surface and used as an absorber. If $|\epsilon(x)\mu(x)|\gg\epsilon_0\mu_0$, so that $(\epsilon/\epsilon_0)(\mu/\mu_0)-\sin^2\theta$ is nearly independent of the incidence angle θ, then the amplitude reflection $R(\theta)$ is wholly determined by $R(0)$. Typical results: When $R(\theta_0)=0$ at one polarization, then at $\theta=\theta_0$ the reflection for the other polarization corresponds to a voltage standing-wave ratio SWR $=\sec^2\theta_0$. At perpendicular polarization max $|R(\theta)|$ on (θ_1, θ_2) is least, for given $|R(0)|$, if $R(0)$ is real and positive; and then $R(\theta)=0$ at $\tan^2\theta/2=R(0)$. But for parallel polarization $R(0)$ must be real and negative to get optimum performance. When the absorber functions at both polarizations the best obtainable result is $|R(\theta)|=\tan^2\theta/2$, no matter what interval (θ_1, θ_2) is specified. The error in the approximation is investigated theoretically and experimentally. A complete set of graphs is included, suitable for design of those absorbers to which the theory applies. The analysis also yields an exact expression for the limiting behavior of the reflection at grazing incidence. This can be used in problems such as computation of the field due to a dipole over a plane earth. Finally, the theory of the Salisbury screen is re-examined as an aid in checking the other developments.**

Statement of the Problem

An important problem in electromagnetic theory is the design of absorbers; that is, surfaces which have zero transmission and small reflection. Often it is desired to have these properties not at one incidence angle θ only, but over a range of angles. Since zero reflection cannot be attained over such a range, we are led to a minimax problem: to minimize the maximum reflection over the range.

For a broad class of absorbers (namely, the *thin solid* absorbers of the present article) this problem was solved nearly six years ago. Though a summary of the results was published at that time [1], continuing interest [2] suggests that the method should be made more generally available. Such is the purpose of this paper.

The mathematical formulation depends on a Riccati equation for the reflection [3]. Let $R=R(\theta)=R(\theta, x)$ denote the complex amplitude reflection when the thickness of the absorber is x, and let $e(x)=\epsilon/\epsilon_0$ and $m(x)=\mu/\mu_0$ denote the complex normalized dielectric constant and permeability at a distance x from the terminating interface (Fig. 1). The main point of the present analysis is to introduce a variable y defined by

$$w=\frac{1+R}{1-R}, \quad y_\perp = w_\perp \sec\theta, \quad y_\| = w_\| \cos\theta. \qquad (1)$$

(Here, as elsewhere in this paper, the subscript $\|$ or $\perp$ specifies the polarization.) In terms of y the Riccati equations are

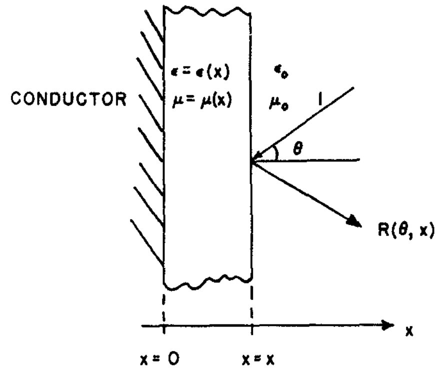

Fig. 1—Stratified medium in cross section.

$$\begin{cases} \dfrac{dy_\perp}{dx} = -\dfrac{2\pi j}{\lambda}\left[\dfrac{me-\sin^2\theta}{m}y_\perp^2 - m\right], \\ \dfrac{dy_\|}{dx} = -\dfrac{2\pi j}{\lambda}\left[ey_\|^2 - \dfrac{me-\sin^2\theta}{e}\right]. \end{cases} \qquad (2)$$

Analysis of $|R(\theta)|$ is the problem with which we are concerned. Subject to an approximation described in the next paragraph, we answer such questions as: What design minimizes the maximum reflection at a given polarization over a given range (θ_1, θ_2), and what is the minimax reflection so attained? What is the resulting reflection at the other polarization? What design minimizes the maximum reflection when this maximum is considered not only with respect to θ, but also with respect to polarization? What is the optimum reflection so obtained? Though approximate, the analysis applies to a wide variety of cases of practical interest. Besides their relevance to the problem of design, the results obtained yield objective criteria by which the performance of any given absorber can be judged. The furnishing of such criteria is an important part of the absorber problem.

The Approximation

If θ ranges from $\theta_1\geq 0$ to θ_2, it is possible to replace $\sin^2\theta$ by a constant in such a way that the maximum error committed does not exceed

$$\tfrac{1}{2}(\sin^2\theta_2 - \sin^2\theta_1).$$

For example, on (0, 25°) the error is not more than 0.09, and on (0, 45°) it is, at most, 0.25. Even on the whole range (0, 90°), the error is ≤ 0.5. This fact suggests the approximation

$$m(x)e(x) - \sin^2\theta \cong m(x)e(x) - \sin^2\theta_0 \qquad (3)$$

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in (2), where θ_0 is a suitably chosen value between θ_1 and θ_2. The value θ_0 is allowed to depend on θ_1 and θ_2, and on x if desired, but not on θ.

The validity of the approximation is investigated later in this paper. For the present, we note that it is surely justified when Re $(me)\gg 1$. Thus, if the true value of Re (me) is 10, the effect of our approximation on (0, 45°) is similar to that of taking Re (me) somewhere between 9.75 and 10.25. Since few artificial dielectrics of the type used in absorbers can be held to such small tolerances, the approximation seems entirely realistic. Indeed, because of these manufacturing tolerances, the absorber is not really a stratified medium at all; and in the author's opinion the modified equations in (2), resulting from (3), are just as appropriate (or inappropriate) as are those in (2) themselves.

If the thickness x is large, the error may build up in the manner characteristic of differential equations; and when Re $(me)\cong 1$, the approximation is also less easy to justify. (We shall see, nevertheless, that the approximation can be excellent in this latter case.) To keep in mind the situation to which our analysis applies for θ far from 0°, the reader may think of a *thin solid* absorber. The word *solid* suggests Re $(me)\gg 1$, which is not the case for low density foams. If $\theta \doteq 0°$, the approximation is valid regardless of the type of absorber considered.

It should be mentioned, in conclusion, that a given accuracy of approximation for $\sin^2\theta$ does not usually insure the same accuracy for y. However, the one error can be estimated in terms of the other. For example, let E, F, and M be constants such that for $0\leq p\leq 1$

$$|e(x)| < E, \quad |1/e(x)| < F, \quad |m(x) - p/e(x)| < M$$

throughout the dielectric material. If y refers to the value for θ and y_0 to the value for θ_0 at parallel polarization, it can be shown that

$$|y - y_0| < \frac{2\pi x}{\lambda} |\sin^2\theta - \sin^2\theta_0| F \sec^2 \frac{2\pi x}{\lambda}\sqrt{ME}. \quad (4)$$

The thickness x of the absorber must be such that the argument of the secant is $<\pi/2$. A limitation of this sort will arise in any estimation of y because $|y|$ can be (and generally is) unbounded. (On the other hand, $|R|$ does not depend critically on y when $|y|$ is large; see the section entitled "The Salisbury Screen.")

The Basic Formula for Reflection

In accordance with (3) let θ in (2) be replaced by θ_0. Since $R=-1$ when $x=0$, the initial conditions are

$$y_\perp = 0, \qquad y_\parallel = 0 \text{ at } x = 0. \quad (5)$$

It is very important that these conditions are independent of θ. This same independence would be observed if $R=+1$ at $x=0$, the short circuit being replaced by an open circuit; and our analysis applies without change to that case. This remark will be needed later.

Since m, e, $1/m$, and $1/e$ are bounded for any physically realizable materials, the right-hand members of (2) satisfy a Lipschitz condition on y. The uniqueness theorem insures that there is only one solution satisfying the conditions (5). Thus, for each fixed x,

$$w_\perp \sec\theta = \text{constant}, \quad w_\parallel \cos\theta = \text{constant}, \quad (6)$$

independent of θ. These are the fundamental relations for a thin solid absorber. If $w(0)=p\exp(-jq)$ we get the formulas

$$R_\perp(\theta) = \frac{pe^{-jq}\cos\theta - 1}{pe^{-jq}\cos\theta + 1}, \qquad R_\parallel(\theta) = \frac{pe^{-jq} - \cos\theta}{pe^{-jq} + \cos\theta}$$

which admit a physical interpretation [2].

Transforming back from $w(0)$ to $R(0)$ yields the following: *Let the normal-incidence reflection have amplitude a and phase b, so that* $R(0)=a\exp(-jb)$. *Then the reflection $R(\theta)$ at incidence θ is wholly determined by a and b.* Indeed, with $t=\tan^2\theta/2$ we have

$$|R_\perp(\theta)|^2 = \frac{t^2 - 2at\cos b + a^2}{1 - 2at\cos b + a^2t^2} \quad (7)$$

at perpendicular polarization and $|R_\parallel(\theta)|^2$ equals the same, with $+\cos b$ instead of $-\cos b$.

Graphical representation is given in Figs. 2–4. Since absorbers are commonly described in terms of their decibel attenuation, we have plotted the absorption

$$A(\theta) = -\log_{10} |R(\theta)|^2 = \text{db down}$$

rather than the power reflection, $|R(\theta)|^2$. In each figure $a=|R(0)|$ is held constant, while the phase b is a parameter.

According to (7) the same family of curves can be used for both polarizations. In fact, *let two absorbers have the same normal-incidence reflection except for* 180° *change in phase. Then one absorber has the same behavior at perpendicular polarization, as the other has at parallel polarization.* This fact is exploited in the figures by appropriate designation of b.

Optimum Design, Fixed Polarization

With freedom to adjust the two arbitrary complex functions $e(x)$ and $m(x)$, we should have expected a wide variety of possible behaviors, $|R|$ vs θ. But the foregoing considerations show that this expectation is sharply revised when thin, solid absorbers are in question. The angular dependence has a rigidity which is quite unlooked for, in view of the generality of the media considered. We present criteria for optimum design, with due regard to this rigidity.

At a fixed polarization let it be required to minimize the maximum reflection over the given range (θ_1, θ_2). The design is carried out by use of Fig. 5, which gives

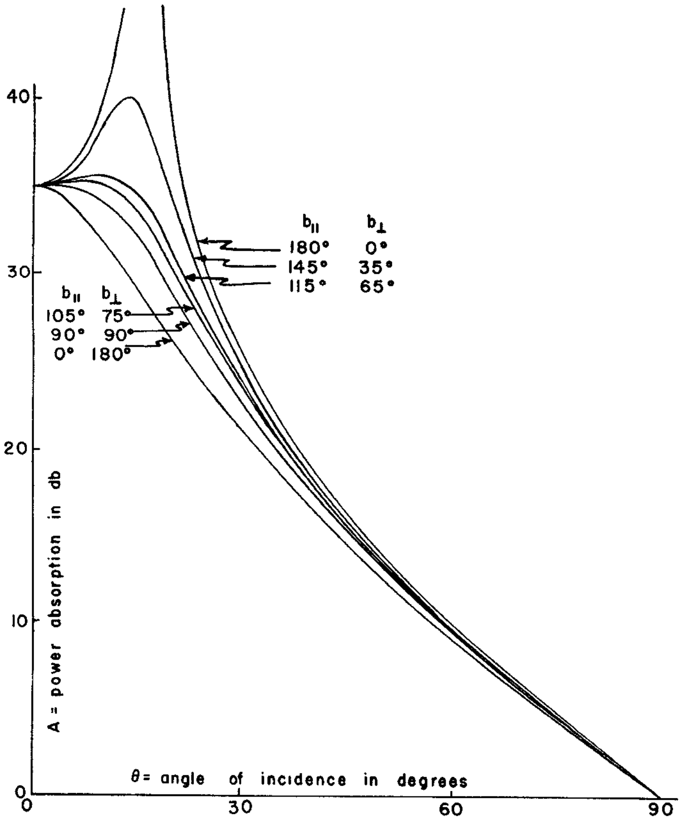

Fig. 2—Power reflection in decibels vs incidence angle when $R(0)=0.0178\ e^{ib}$.

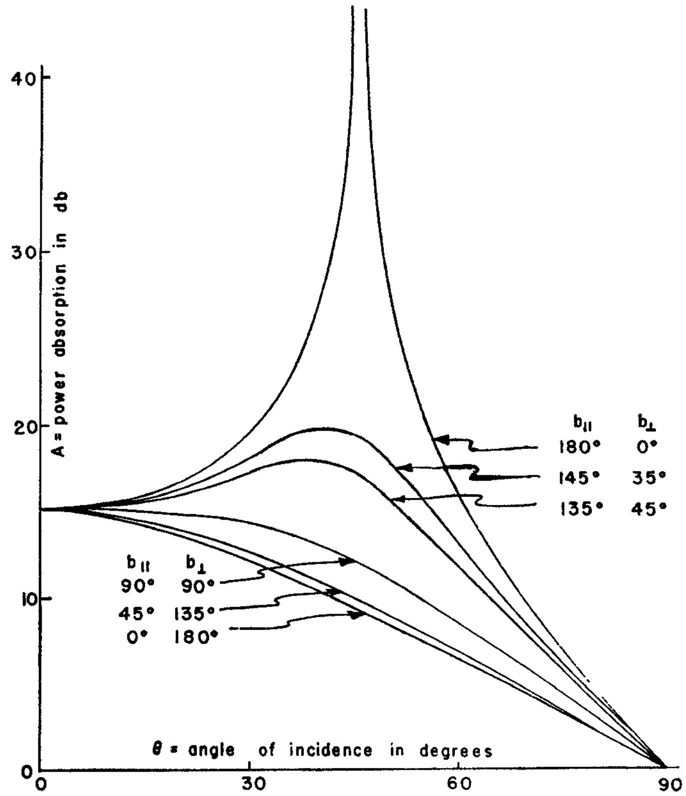

Fig. 4—Same, $R(0)=0.178\ e^{ib}$.

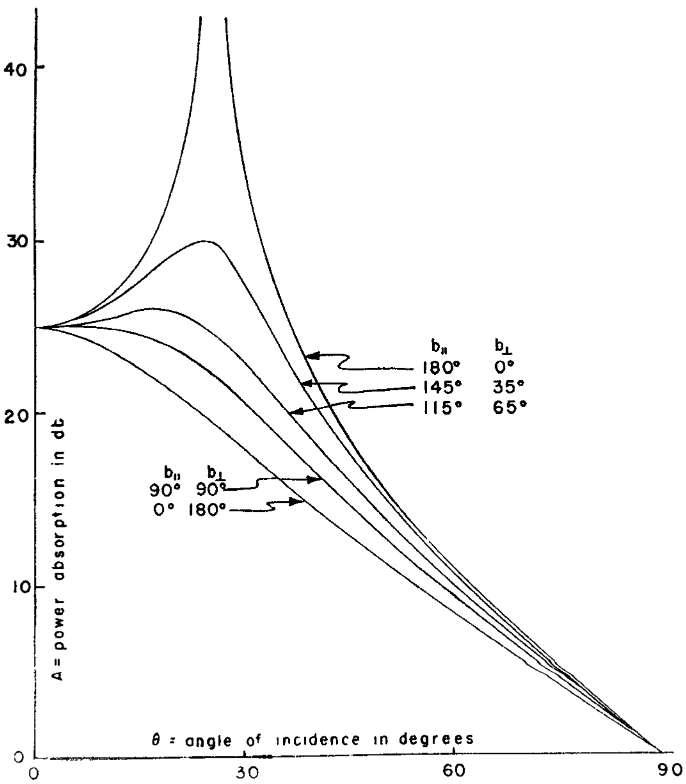

Fig. 3—Same, $R(0)=0.0560\ e^{ib}$.

contours of constant reflection for optimum design vs the incidence angle θ and the design angle, ϕ, at which $R(\theta)=0$. The optimum choice of ϕ is the unique value such that

$$\theta_1 \leq \phi \leq \theta_2 \quad \text{and} \quad A(\theta_1) = A(\theta_2) \tag{8}$$

on the corresponding contour. By considering horizontal line segments extending from θ_1 to θ_2, one readily establishes the ϕ at which (8) holds. Such a segment is shown in the figure for the range $30° < \theta < 60°$. It gives $\phi = 49°$ and the minimax absorption is about 17 db.

The optimum absorber is specified as soon as $R(0) = ae^{-ib}$ is known. Elementary analysis shows that

$$a = \tan^2 \phi/2, \qquad b_\perp = 0°, \qquad b_\parallel = 180°. \tag{9}$$

In summary: *If a thin, solid absorber gives optimum performance for $\theta_1 < \theta < \theta_2$ at a given polarization, then the normal-incidence reflection must have zero phase shift when the given polarization is perpendicular and 180° phase shift when it is parallel. The optimum absorber and its performance are given by the relation plotted in Fig. 5, together with* (9).

Here, the absorber is unique only insofar as its behavior is determined by $R(0)$. Many choices of $\epsilon(x)$ and $\mu(x)$ are possible.

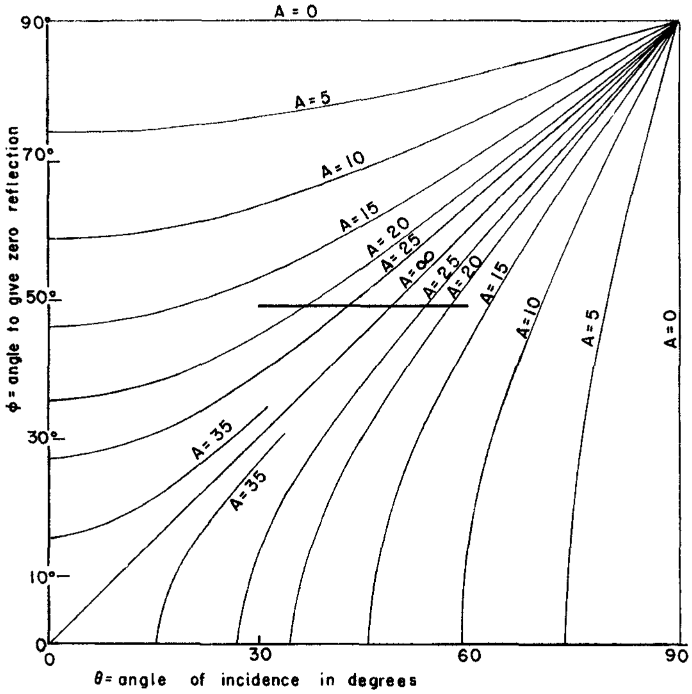

Fig. 5—Contours of constant power reflection in decibels, for a given polarization, vs the incidence angle θ and the design angle ϕ at which the reflection is zero.

Change of Polarization

We have seen that $|R(\theta)|$ must have a minimum between θ_1 and θ_2 if the design is to be optimum at a given polarization. At the other polarization $|R(\theta)|$ cannot have any minimum between 0 and $\pi/2$; indeed, the basic formula readily gives the following: *For a thin, solid absorber, a necessary and sufficient condition that the amplitude reflection* $|R(\theta)|$ *have a minimum at* $\theta=\phi\neq 0$ *is that* $R(\phi)$ *be pure imaginary. This situation arises at perpendicular polarization if, and only if,* $|w(0)|>1$ *(or if* $-\pi/2<\angle R(0)<\pi/2$*, which is the same thing). It arises at parallel polarization in the contrary case.*

Thus, a minimum between 0 and $\pi/2$ [hence, very good performance on (θ_1, θ_2)] is possible for one polarization only, not for both with a given absorber. More detailed analysis yields the following. *For a thin, solid absorber, let the amplitude reflection* $|R(\theta)|$ *have a minimum equal to* tan $q/2$ *at* $\theta=\phi\neq 0$*. Then the reflection is wholly determined at both polarizations, and at arbitrary incidence, by* ϕ *and* q. We have

$$|R(\theta)|^2=\frac{\cos^2\phi-2\cos\phi\cos\theta\cos q+\cos^2\theta}{\cos^2\phi+2\cos\phi\cos\theta\cos q+\cos^2\theta} \qquad (10)$$

at the given polarization, and the same, with sec θ replacing cos θ, at the other. In particular, suppose $R(\theta)=0$ at one polarization. Then for the standing-wave ratio at the other polarization

$$\text{SWR}\equiv\frac{1+|R(\phi)|}{1-|R(\phi)|}=\sec^2\phi. \qquad (11)$$

Despite these negative results, an absorber may (and generally does) have to perform at both polarizations simultaneously. Since $|R_\perp|$ increases and $|R_\parallel|$ decreases as b increases from 0 to 180°, the value of b which is best for one polarization is worst for the other. We have $|R_\perp|=|R_\parallel|$ if, and only if, $b=\pi/2$; and for optimum design

$$\max(|R_\perp|, |R_\parallel|) = |R_\perp| = |R_\parallel| = \left(\frac{t^2+a^2}{1+t^2a^2}\right)^{1/2}, \qquad (12)$$

where $t=\tan^2\theta/2$. The expression, which is plotted in Fig. 6, increases with a and hence is least at $a=0$. These results may be summarized as follows: *Suppose a thin, solid absorber, intended for use at both polarizations, has prescribed normal incidence reflection, a. Then for any range of* θ *the design is optimum* [*subject to* $|R(0)|=a$] *if and only if* $R(0)$ *is pure imaginary. In that case the reflection is independent of polarization and is given by the relation represented graphically in Fig. 6. The reflection at every angle decreases as* $|R(0)|$ *decreases, and when* $R(0)=0$ *we have* $|R(\theta)|=\tan^2\theta/2$*. This represents the optimum performance possible.*

Experimental Verification

Comparison of theory and experiment leads to satisfactory agreement. The absorbers used were the standard production model F-89-VF, supplied by the McMillan Laboratory, Inc., Ipswich, Mass., where the measurements were made by A. Preston and the author. The theoretical curve showed that the absorbers have approximately the optimum behavior possible (according to our theory) in the range for which they were designed; *viz.*, at perpendicular polarization, and over an interval centered near $\theta=35°$. The constants (a, b) were determined, in fact, by taking $R_\perp(35°)=0$ and using (10). The curve for parallel polarization was determined by the same choice of a and b; it agreed within 3 db out to the last data-point, $\theta=60°$. Since highly accurate data have been presented elsewhere [2] we do not devote much space to the subject here. The point to be emphasized is that the absorbers had a rather complicated internal structure and that *the theory describes their behavior without taking account of this structure.*

Grazing Incidence

The foregoing methods lead to the following: Let $\alpha=\alpha(x)$ satisfy the Riccati equation and boundary condition

$$\frac{d\alpha}{dx}=\frac{-2\pi j}{\lambda}\left[\frac{m(x)e(x)-1}{m(x)}\alpha^2-m(x)\right],\quad \alpha(0)=0. \qquad (13)$$

Let $R_\perp(\theta)$ be the reflection at perpendicular polarization and incidence θ for a medium of thickness x and complex parameters $\epsilon/\epsilon_0=e(x)$, $\mu/\mu_0=m(x)$, backed by a con-

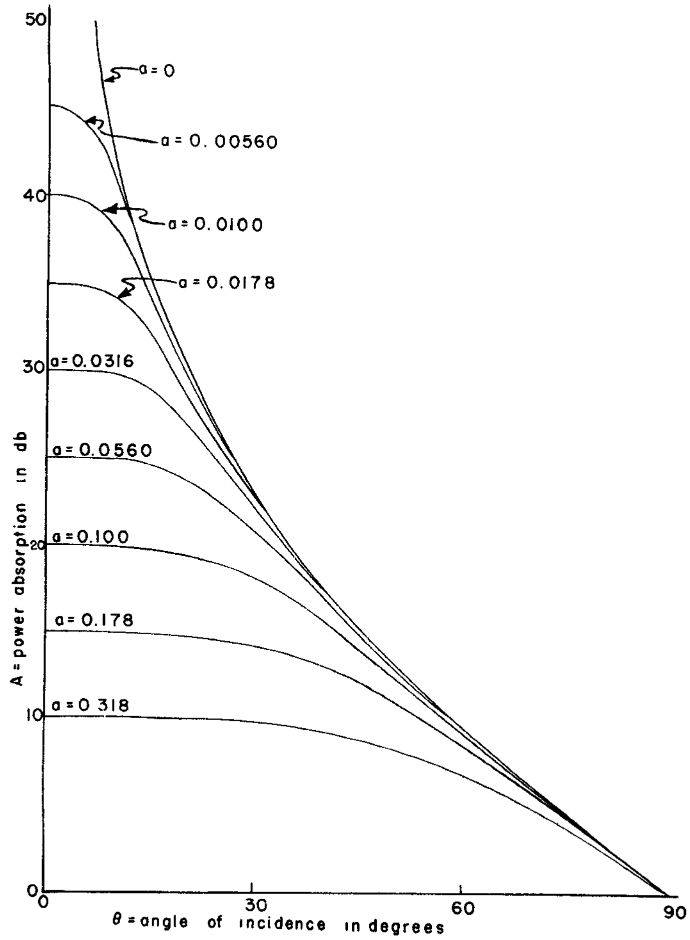

Fig. 6—Power reflection in decibels vs the incidence angle θ, when max $(|R_\perp|, |R_\parallel|)$ is minimized by suitable choice of arg $R(0)$.

ducting surface. Then

$$\lim_{\theta\to\pi/2} \frac{1 + R_\perp}{\cos\theta} = 2\alpha$$

and

$$\frac{d}{d\theta} |R_\perp|^2 = 4 \operatorname{Re}(\alpha) \text{ at } \theta = \pi/2. \tag{14}$$

Similarly, if $\beta = \beta(x)$ satisfies

$$\frac{d\beta}{dx} = \frac{-2\pi j}{\lambda}\left[e(x)\beta^2 - \frac{m(x)e(x) - 1}{e(x)}\right], \quad \beta(0) = 0 \tag{15}$$

then the complex reflection at parallel polarization satisfies

$$\lim_{\theta\to\pi/2} \frac{1 - R_\parallel}{\cos\theta} = \frac{2}{\beta},$$

$$\frac{d}{d\theta} |R_\parallel|^2 = \operatorname{Re}\left(\frac{4}{\beta}\right) \text{ at } \theta = \pi/2. \tag{16}$$

It should be emphasized that (13)–(16) are exact; that is, they follow from (2) without the approximation (3) used heretofore in this discussion.

The relevance to the problem of θ dependence is as follows. Eqs. (14) and (16) are consequences of the fact that

$$\lim w_\perp \sec\theta = \alpha, \quad \lim w_\parallel \cos\theta = \beta \text{ as } \theta \to 90°. \tag{17}$$

If we take $w \cos\theta = \beta$, we get the approximate formula

$$|R_\parallel|^2 \doteq \frac{P^2 - 2P\cos\theta\cos Q + \cos^2\theta}{P^2 + 2P\cos\theta\cos Q + \cos^2\theta} \tag{18}$$

upon setting $\beta = Pe^{iQ}$. This result plays the same role for $\theta \doteq 90°$ as the previous results do for $\theta \doteq 0°$. A similar approximation is true for $R_\perp$.

Since (13) and (15) can be made to yield any desired α and β by appropriate choice of m and e, the results for perpendicular and parallel polarization are independent (in contrast to the behavior near 0°, where they were dependent). Apart from this, the previous discussion and graphs apply here too; for (17) has the same structure as (6).

The Salisbury Screen

An excellent check of this theory is afforded by the Salisbury screen (Fig. 7). If the resistive layer has thickness t_1 and complex parameters

$$e_1 = k_1(1 - j\tan\delta_1), \qquad m_1 = k_1'(1 - j\tan\delta_1'),$$

the reader will recall that its transmission T_1 and reflection R_1 are readily computed under the hypothesis

$$|t_1\sqrt{m_1e_1}| \ll \lambda, \qquad |e_1/m_1| \gg 1, \qquad |e_1m_1| \gg 1.$$

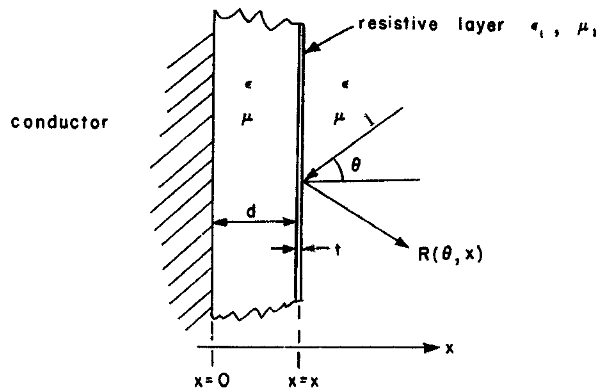

Fig. 7—Salisbury screen in cross section.

The result is

$$T_1 \doteq 1 + R_1 \doteq [1 + \tfrac{1}{2}KN]^{-1} \tag{19}$$

where N is the normalized conductivity and $K = \sec\theta$, or $K = \cos\theta$, at perpendicular, and at parallel polarization, respectively.

If the lossless dielectric separating the resistive and conducting layers has thickness d and real parameter $k = (\epsilon/\epsilon_0)(\mu/\mu_0)$, the reflection of core-plus-metal is very close to exp $(-2j\psi)$ where

$$\psi = 2\pi d\sqrt{k - \sin^2\theta}/\lambda. \tag{20}$$

(There is no error in (20) when $k=1$ or when ψ is a multiple of $\pi/2$. This fact will be useful later.)

Without further approximation we get

$$y_\perp = [N - j\cot\psi\cos\theta]^{-1}, \quad y_\parallel = [N - j\cot\psi\sec\theta]^{-1} \quad (21)$$

for the Salisbury screen. If the reflection is 0 at $\theta=\phi$, then

$$\psi = \frac{\pi}{2}\sqrt{\frac{k-\sin^2\theta}{k-\sin^2\phi}}, \quad N_\perp = \cos\phi, \quad N_\parallel = \sec\phi \quad (22)$$

for a first-order, *i.e.*, thinnest possible, structure. The resulting behavior at incidence θ is given by (21). Taking $\phi=0$ as an important and typical case, we get

$$y_\perp = \left[1 - j\cos\theta\cot\frac{\pi}{2}(1-k^{-1}\sin^2\theta)^{1/2}\right]^{-1}$$

$$y_\parallel = \left[1 - j\sec\theta\cot\frac{\pi}{2}(1-k^{-1}\sin^2\theta)^{1/2}\right]^{-1}. \quad (23)$$

When $k=1$, the real and imaginary parts are given in Table I as a function of θ. For perpendicular polarization the variation is not excessive, but for parallel polarization the real part varies from 1 to 0. A graph of the real part vs the imaginary part yields very nearly the same curve both times; but the whole curve obtained for perpendicular polarization on (0, 90°) is traced out by the parallel polarization values on (0, 44°). Thus, for our case $|e_1| \gg |m_1|$, the theory for $R_\perp$ is more reliable. just as the experimental work suggests. (If $|m_1| \gg |e_1|$, the theory for $R_\parallel$ is the better.)

According to the general theory

$$|R|^2 = \tan^4\theta/2 \quad (24)$$

at both polarizations when, as in this case, $R(0)=0$. To see how serious an error in $|R|$ is produced by the variation shown in Table I, we have plotted the reflection given by (23) for $k=1$ together with that given by (24).

By Fig. 8 the maximum error is about 5 db. In absorber design, an error of 5 db is not as serious as one might think, so that the theory is not wholly vitiated even by the great variation shown in Table I.

If a larger value for k is chosen, the assumption of a *thin solid* absorber is better satisfied, and we expect a smaller error. Table II presents the same calculation as Table I, except that $k=5$. The basic relation (6) is satisfied very accurately at perpendicular polarization on the whole range (0, 90°), and it is satisfied to 70° or 80° for the parallel case. Since the real part will always change from 1 to 0 in the latter case, the equality over the whole range is not possible. However, the effect on $|R|$ is completely negligible even when $\theta \doteq 90°$, as we shall see. The analog of Fig. 8 for Table II leads to three curves that are practically indistinguishable.

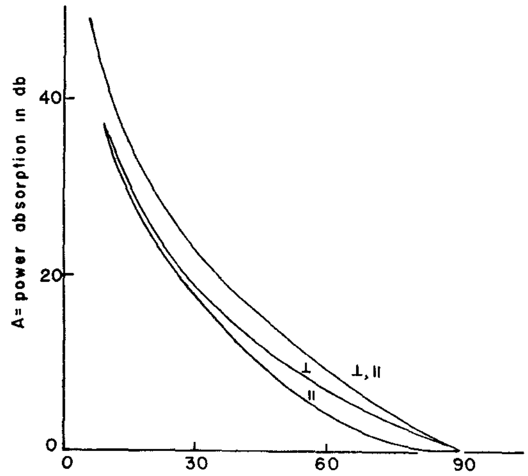

Fig. 8—Comparison of the general theory with the Salisbury screen theory when the core dielectric constant $k=1$ and the resistivity $\Omega=377$ ohms.

TABLE I

SALISBURY SCREEN IMPEDANCE FOR $k=1$, $N=1$, $d/\lambda=\frac{1}{4}$

θ (degrees)	0	10	20	30	40	50	60	70	80	90
Re ($w_\perp\sec\theta$)	1.0000	0.9994	0.992	0.97	0.92	0.86	0.80	0.75	0.72	0.71
Im ($w_\perp\sec\theta$)	0	0.024	0.090	0.18	0.27	0.34	0.40	0.43	0.45	0.45
Re ($w_\parallel\cos\theta$)	1.0000	0.9994	0.990	0.95	0.80	0.51	0.20	0.04	0.002	0
Im ($w_\parallel\cos\theta$)	0	0.024	0.10	0.22	0.40	0.50	0.40	0.20	0.04	0

TABLE II

SALISBURY SCREEN IMPEDANCE FOR $k=5$, $N=1$, $d\sqrt{5}/\lambda=\frac{1}{4}$

θ (degrees)	0	10	20	30	40	50	60	70	80	90
Re ($w_\perp\sec\theta$)	1.0000	0.99998	0.9997	0.9989	0.9972	0.9966	0.9975	0.9976	0.9992	1.000
Im ($w_\perp\sec\theta$)	0	0.0046	0.0713	0.034	0.053	0.059	0.062	0.049	0.028	0
Re ($w_\parallel\cos\theta$)	1.0000	0.99998	0.9996	0.9980	0.9929	0.980	0.942	0.858	0.535	0
Im ($w_\parallel\cos\theta$)	0	0.0048	0.020	0.045	0.090	0.14	0.23	0.36	0.50	0

Further Verification of the General Theory

If $R=0$ at $\theta=\phi$, (21) and (22) yield

$$R(\phi) = \pm \frac{1-\cos^2\phi}{1+\cos^2\phi} \tag{25}$$

for the reflection at the other polarization. This agrees with (11), so that one major prediction of the theory is precisely verified. But it should be emphasized that (25) follows with no approximation other than (19); in particular, $k \gg 1$ for the core is *not* assumed.[1]

The reason for the success of the theory, even though $k \doteq 1$, is as follows. It happens that core-plus-metal is an exact open circuit at both polarizations, in the circumstances leading to (25). Since the cloth does not know whether this open circuit at $\theta=\phi$ is produced by a low or by a high dielectric constant, we can replace the core by a thinner one having $k \gg 1$ without changing the reflection at the angle in question. But the new absorber satisfies our assumptions, since the cloth does, and since, as we were saying, $k \gg 1$. Thus, its reflection can be computed by (11).

The reader will perceive that we have arrived at a general principle: *Let a thin solid layer, A, be backed by a terminating stratified medium, B, and suppose the over-all reflection is zero at a given angle ϕ and polarization. If there is a thin solid absorber whose complex reflection reproduces that of B at ϕ and at both polarizations, then* (11) *holds for the original composite medium, A plus B.* A mathematical proof of this principle can be based on certain functional equations satisfied by solutions of Riccati's equation, but the physics is so clear that we do not belabor the matter here.

A final check of the theory is given by letting $\theta \to 0$ or $\theta \to 90°$. By (21)

$$|R_\perp|^2 \sim |R_\parallel|^2 \sim [1+(\pi/2k)^2]\tan^4\frac{\theta}{2}$$

as $\theta \to 0$, where we write $a \sim b$ to mean $\lim a/b = 1$. Comparing with (24) we see that the general theory is in error by the factor $1+(\pi/2k)^2$, as $\theta \to 0$. For $k=1$ this accounts for the difference of 5.4 db that occurs in Fig. 8. For $k=2$ the error due to this factor is 2.1 db, and it is 1 db for $k=3$.

Since the theory gives the correct result near 0°, the first terms of the Taylor's series for $|R|^2$ must agree, though there is no guarantee of equality of limiting ratios when $R(0)=0$. In the present case the agreement is exact through terms in θ^3, and the ratio of coefficients for θ^4 tends rapidly to 1 as $k \to \infty$.

A similar calculation for θ near 90° gives

$$1 - |R_\perp|^2 \sim 4\cos\theta$$

$$1 - |R_\parallel|^2 \sim 4\left(1+\cot^2\frac{\pi}{2}(1-k^{-1})^{1/2}\right)^{-1}\cos\theta$$

whereas by (24)

$$1 - |R|^2 \sim 4\cos\theta$$

at either polarization. Expansion of the radical in powers of $1/k$ yields

$$1 - |R_\parallel|^2 \sim 4[1+(\pi/4k)^2]^{-1}\cos\theta.$$

Thus, as $\theta \to 90°$ the theory yields the correct behavior for perpendicular polarization regardless of k, and the correct behavior for parallel polarization provided $(\pi/4k)^2 \ll 1$. The theory underestimates the value of $|R_\parallel|$. However, the error in $1-|R_\parallel|^2$ is only 1 db for $k=1.5$ and for $k=5$ as in Table II, the error is only 0.011 db. In practice the important thing is $|R|$, not w; and that Re $(w_\parallel \cos\theta)$ jumps from 0.858 to 0 on the range (70°, 90°) in Table II is without practical significance.

[1] Eq. (25) was noted by Walther [2]. However, he assumes that core-plus-metal has an electrical quarter-wave thickness independently of θ. By (20) this is equivalent to $k \gg \sin^2\theta$; so that Walther's observation is a direct consequence of the general theory. For the same reason, his analysis of the Salisbury screen does not enable us to compute Tables I and II.

References

[1] Abstract 158t, *Bull. Amer. Math. Soc.*, vol. 60, p. 79; January, 1954. In 1957 the results of sections 3–5 were communicated orally to H. Severin and his co-workers at the Physikalisches Institut der Universität Göttingen.

[2] K. Walther, "Polarisations- und Winkelabhängigkeit des Reflexionsfaktors, von Absorbern für elektromagnetische Zentimeterwellen," *Z. angew. Phys.*, vol. 10, pp. 285–295; June 6, 1958.

[3] R. B. Barrar and R. M. Redheffer, "On nonuniform dielectric media," IRE Trans. on Antennas and Propagation, vol. AP-3, pp. 101–107; July, 1955.

[4] R. Pottel, "Über der Erhöhung der Frequenzbandbreite dünner λ/4-Schicht Absorber für elek. Zentimeterwellen," *Z. angew. Phys.*, vol. 11, pp. 46–51; February, 1959.

[5] H. Haddenhorst, "Durchgang von elektromagnetischen Wellen durch inhomogene Schichten (Teil II: Absorbtion), *Z. angew. Phys.*, vol. 8, pp. 264–267; June, 1956.